ESD-TR-71-192



# Technical Note 1971-32

# A New Geometrical Theorem Discovered with the Aid of a Computer

W. L. Black
H. C. Howland
B. Howland

21 June 1971

Prepared under Electronic Systems Division Contract F19628-70-C-0230 by

**Lincoln Laboratory**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

# A NEW GEOMETRICAL THEOREM DISCOVERED WITH THE AID OF A COMPUTER

*W. L. BLACK*

*Division 6*

*H. C. HOWLAND*

*Cornell University*
*Division of Biological Sciences*
*Ithaca, New York 14850*

*B. HOWLAND*

*Group 63*

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LEXINGTON MASSACHUSETTS

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract F19628-70-C-0230.

## ABSTRACT

In the course of a computer study of a new form of ball bearing, a curious invariance was noted. This led to a new theorem in the geometry of circles. A proof for this theorem, together with a useful lemma, is the subject of this Technical Note.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
Chief, Lincoln Laboratory Project Office

A NEW GEOMETRICAL THEOREM DISCOVERED WITH THE AID OF A COMPUTER

Let $\Gamma$ and $\tilde{\Gamma}$ be two circles in euclidean 3-space, $R^3$. Suppose there is a number $x$ such that: (1) every point on either circle is distance $x$ from exactly two points on the other circle. We can then select a point $Z_1$ on $\Gamma$ and draw a zig-zag line between the two circles as follows:

$$
\begin{array}{llllll}
\text{Select} & \tilde{Z}_1 & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_1 - Z_1| = x \\
\text{Select} & Z_2 \neq Z_1 & \text{on} & \Gamma & \text{with} & |Z_2 - \tilde{Z}_1| = x \\
\text{Select} & \tilde{Z}_2 \neq \tilde{Z}_1 & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_2 - Z_2| = x \\
 & \cdot & & & & \\
 & \cdot & & & & \\
 & \cdot & & & & \\
 & \cdot & & & & \\
\text{Select} & Z_{n+1} \neq Z_n & \text{on} & \Gamma & \text{with} & |Z_{n+1} - \tilde{Z}_n| = x \\
\text{Select} & \tilde{Z}_{n+1} \neq \tilde{Z}_n & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_{n+1} - Z_{n+1}| = x \quad .
\end{array}
$$

It may happen (illustrated for the case $n = 3$ in Fig. 1) that $Z_{n+1} = Z_1$. We show that if this occurs, the zig-zag line can be started at <u>any</u> point on $\Gamma$ and it will still close.

This remarkable fact was observed while performing certain calculations about ball bearings on a computer[1]. The theorem bears a superficial resemblance to Steiner's Porism[2] but cannot be proved the same way.

Condition (1) is not as formidable as it may appear. If both circles lie in the same plane, with radii $r$ and $\bar{r}$ and with centers separated by $\delta$, elementary calculus shows that (1) is equivalent to the two inequalities:

$$|r - \bar{r}| < x - \delta$$

$$x + \delta < r + \bar{r}$$

Thus suitable $x$'s will exist provided the smaller circle encloses the center of the larger one.

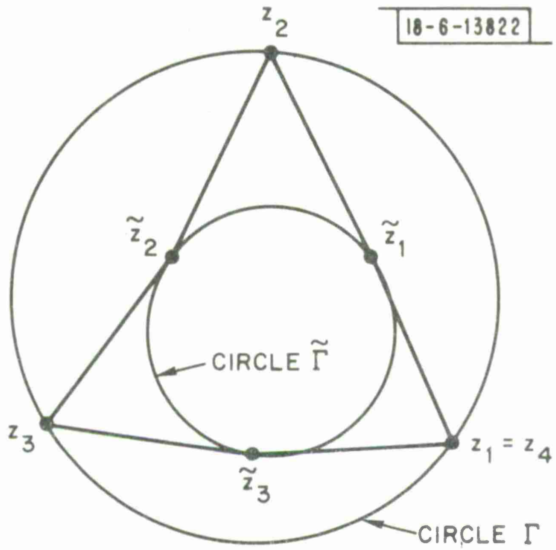


Fig. 1. Each straight line has length x, n = 3.

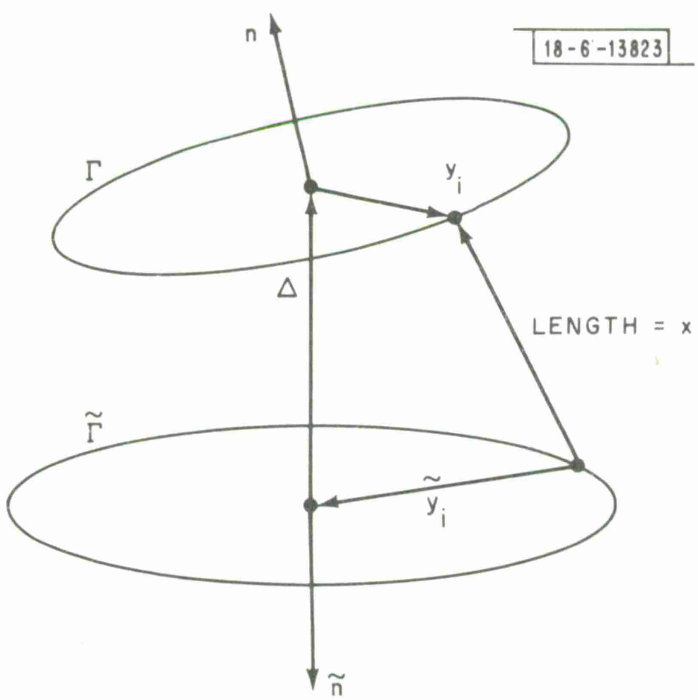


Fig. 2. Illustrating the notation used in the computation of f.

The proof proceeds as follows: Suppose for some choice of $Z_1$, say $Z_1 = p$, $Z_{n+1} = Z_1$. We parametrize $\Gamma$ by $s$, the directed arc length measured from $p$. Thus we can view $Z_1$, and thus $Z_{n+1}$, and ultimately $t$, the arc length from $p$ to $Z_{n+1}$ as functions of $s$. Below we show that there is a smooth function $f$ of $s$ and $t$ with properties:

$$\text{(i.)} \quad f(s,s) = 1 \quad \text{for all } s$$

$$\text{(ii.)} \quad f(s,t) = \frac{dt}{ds}$$

Application of a well known uniqueness theorem[3] assures us that this ordinary differential equation:

$$\frac{dt}{ds} = f(s,t)$$

$$t(o) = 0$$

has only one solution. By (i), $t(s) = s$ is the solution. But this implies that $Z_{n+1} = Z_1$ for all $Z_1$.

In constructing $f$ we will need the following lemma:

<u>Lemma</u>: Let $B$, $n$, $Z_1$ and $Z_2$ be vectors in $R^3$, satisfying:

$$\text{(a.)} \quad |y_1| = |y_2|$$

$$\text{(b.)} \quad |B + y_1| = |B + y_2|$$

$$\text{(c.)} \quad n \cdot y_1 = n \cdot y_2$$

$$\text{(d.)} \quad y_1 \neq y_2$$

Then:

$$B \times n \cdot (y_1 + y_2) = 0 \quad .$$

<u>Proof</u>: If $B$ and $n$ are dependent the result is obvious. If not, by squaring (b) and (a) and taking the difference we obtain:

$$\text{(e.)} \quad B \cdot y_1 = B \cdot y_2$$

In conjunction with (c), (e) shows that $y_1$ and $y_2$ have the same orthogonal projection on the plane spanned by B and n. By (a) and (d) the components of $y_1$ and $y_2$ normal to this plane must be equal and opposite, so $y_1 + y_2$ lies in the B,n plane. From this the conclusion is evident.

We can now compute f. We will use the following notation:

$n$ is the unit normal to circle $\Gamma$

$\tilde{n}$ " " " " $\tilde{\Gamma}$

$y_i$ is the vector from the center of $\Gamma$ to $Z_i$

$\tilde{y}_i$ is the negative of the vector from the center of $\tilde{\Gamma}$ to $\tilde{Z}_i$ .

$\Delta$ is the vector from the center of $\tilde{\Gamma}$ to the center of $\Gamma$ .

Figure two illustrates this notation. Imagine a slight motion of $y_1$ along the circle. Since $dy_1$ is perpendicular to both $n$ and $y_1$ we can write:

$$dy_1 = n \times \frac{y_1}{|y_1|} |dy_1| \tag{2}$$

As $y_1$ moves, $\tilde{y}_1$ also must move, keeping

$$|\Delta + y_1 + \tilde{y}_1| = x \tag{3}$$

Squaring (3) and differentiating gives:

$$(\Delta + y_1 + \tilde{y}_1) \cdot (dy_1 + d\tilde{y}_1) = 0 \tag{4}$$

Substituting into (4), (2) and (5) where:

$$d\tilde{y}_1 = \tilde{n} \times \frac{\tilde{y}_1}{|y_1|} |d\tilde{y}_1| \tag{5}$$

gives:

$$(\Delta + \tilde{y}_1) \cdot n \times \frac{y_1}{|y_1|} |dy_1| + (\Delta + y_1) \cdot \tilde{n} \times \frac{\tilde{y}_1}{|\tilde{y}_1|} |d\tilde{y}_1| = 0 \quad . \tag{6}$$

Thus:

$$\frac{|d\tilde{y}_1|}{|dy_1|} = - \frac{|\tilde{y}_1| \; (\Delta + \tilde{y}_1) \cdot n \times y_1}{|y_1| \; (\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad . \tag{7}$$

Similarly,

$$\frac{|dy_2|}{|d\tilde{y}_1|} = - \frac{|y_2| \; (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1}{|\tilde{y}_1| \; (\Delta + \tilde{y}_1) \cdot n \times y_2} \tag{8}$$

Multiplying (7) by (8) and using the lemma in the forms:

$$(\Delta + \tilde{y}_1) \cdot n \times y_1 = - (\Delta + \tilde{y}_1) \cdot n \times y_2$$

$$(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1 = - (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2$$

gives:

$$\frac{|dy_2|}{|dy_1|} = \frac{(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad . \tag{9}$$

Multiplying $n$ similar expressions gives:

$$\frac{dt}{ds} = \frac{|dy_{n+1}|}{|dy_1|} = \frac{(\Delta + y_{n+1}) \cdot \tilde{n} \times \tilde{y}_{n+1}}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} = f(s,t)$$

Clearly if $Z_{n+1} = Z_1$, then $y_{n+1} = y_1$ and $\tilde{y}_{n+1} = \tilde{y}_1$, whence:

$$f(s,s) = 1 \quad .$$

This completes the proof.

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1. Quarterly Progress Report No. 101, Research Laboratory of Electronics, Massachusetts Institute of Technology, p. 268 (15 April 1971).

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## ACKNOWLEDGMENT


We wish to thank Mr. Fred Zimnoch for his programming assistance.

## DOCUMENT CONTROL DATA – R&D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T. | 2a. REPORT SECURITY CLASSIFICATION Unclassified<br>2b. GROUP None |
| 3. REPORT TITLE A New Geometrical Theorem Discovered with the Aid of a Computer | |
| 4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical Note | |
| 5. AUTHOR(S) *(Last name, first name, initial)* Black, William L.   Howland, Howard C.   Howland, Bradford | |

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| 21 June 1971 | 12 | 3 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO. F19628-70-C-0230 | 9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1971-32 |
| b. PROJECT NO. 649L<br>c.<br>d. | 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-71-192 |
| 10. AVAILABILITY/LIMITATION NOTICES Approved for public release; distribution unlimited. | |
| 11. SUPPLEMENTARY NOTES None | 12. SPONSORING MILITARY ACTIVITY Air Force Systems Command, USAF |

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In the course of a computer study of a new form of ball bearing, a curious invariance was noted. This led us to a new theorem in the geometry of circles. A proof for this theorem, together with a useful lemma, is the subject of this Technical Note.

14. KEY WORDS

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|---|---|
| geometry of circles | lemma |
| theorem | computer study |